Di Ning

Quiz 3
VECTOR CALCULUS
MATH 21D, Sect 002, Winter Quarter, 2013
INSTRUCTOR: Blake Temple

1. (10pts) In the following exercise, find

   a. The mass of the solid.

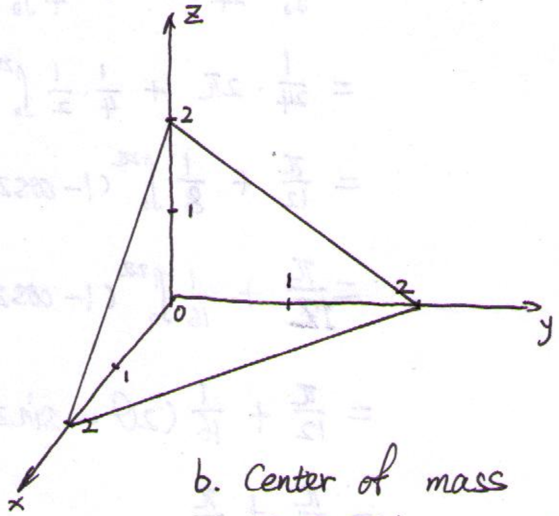
   b. The center of mass.

   A solid region in the first octant is bounded by the coordinate planes and the plane $x+y+z=2$. The density of the solid is $\delta(x,y,z)=2x$.

2. (10pts) Evaluate the cylindrical coordinate integral in the following exercise.

$$\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta+z^2\right)dz\,r\,dr\,d\theta$$

1. Solution.



a. $\text{Mass} = \iiint_V \delta(p)\,dV$

2pts $\boxed{= \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2x\,dz\,dy\,dx}$

$= \int_0^2\int_0^{2-x} 2x(2-x-y)\,dy\,dx$

$= \int_0^2\int_0^{2-x} (4x-2x^2-2xy)\,dy\,dx$

$= \int_0^2 (4xy-2x^2y-xy^2)\Big|_0^{2-x}\,dx$

$= \int_0^2 [4x(2-x)-2x^2(2-x)-x(2-x)^2]\,dx$

x^2-4x+4

$= \int_0^2 [8x-4x^2-4x^2+2x^3-x^3+4x^2-4x]\,dx$

$= \int_0^2 [4x-4x^2+x^3]\,dx$

$= 2x^2-\frac{4}{3}x^3+\frac{1}{4}x^4\Big|_0^2$

$= 8-\frac{4}{3}\times 8+4$

$= \frac{36}{3}-\frac{32}{3} = \boxed{\frac{4}{3}}$ 2pts

b. Center of mass

$\bar{x} = \dfrac{\int_0^2\int_0^{2-x}\int_0^{2-x-y} x\cdot(2x)\,dz\,dy\,dx}{\int_0^2\int_0^{2-x}\int_0^{2-x-y} 2x\,dz\,dy\,dx}$ (1pt)

$= \frac{16}{15}\Big/\frac{4}{3} = \frac{4}{5}$ (1pt)

$\bar{y} = \dfrac{\int_0^2\int_0^{2-x}\int_0^{2-x-y} y\cdot(2x)\,dz\,dy\,dx}{\int_0^2\int_0^{2-x}\int_0^{2-x-y}(2x)\,dz\,dy\,dx}$ (1pt)

$= \frac{8}{15}\Big/\frac{4}{3} = \frac{2}{5}$ (1pt)

$\bar{z} = \dfrac{\int_0^2\int_0^{2-x}\int_0^{2-x-y} z(2x)\,dz\,dy\,dx}{\int_0^2\int_0^{2-x}\int_0^{2-x-y}(2x)\,dz\,dy\,dx}$ (1pt)

$= \frac{8}{15}\Big/\frac{4}{3} = \frac{2}{5}$ (1pt)

2. Solution:

$$\int_0^{2\pi}\int_0^1\int_{-\frac{1}{2}}^{\frac{1}{2}} (r^2\sin^2\theta + z^2)\, dz \cdot r \cdot dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^2\sin^2\theta\, z + \tfrac{1}{3}z^3)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} \cdot r \cdot dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^2\sin^2\theta + \tfrac{1}{3}\cdot\tfrac{1}{4})\, r\, dr\, d\theta \quad (2pts)$$

$$= \int_0^{2\pi} (\tfrac{1}{4}r^4\sin^2\theta + \tfrac{1}{24}r^2)\Big|_0^1\, d\theta$$

$$= \int_0^{2\pi} (\tfrac{1}{4}\sin^2\theta + \tfrac{1}{24})\, d\theta \quad (2pts)$$

$$= \int_0^{2\pi} \tfrac{1}{24}\, d\theta + \tfrac{1}{4}\int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= \tfrac{1}{24}\cdot 2\pi + \tfrac{1}{4}\cdot\tfrac{1}{2}\int_0^{2\pi} (1-\cos 2\theta)\, d\theta$$

$$= \tfrac{\pi}{12} + \tfrac{1}{8}\int_0^{2\pi} (1-\cos 2\theta)\cdot\tfrac{1}{2}\cdot d(2\theta)$$

$$= \tfrac{\pi}{12} + \tfrac{1}{16}\int_0^{2\pi} (1-\cos 2\theta)\, d(2\theta) \quad (1pt)$$

$$= \tfrac{\pi}{12} + \tfrac{1}{16}(2\theta - \sin 2\theta)\Big|_0^{2\pi}$$

$$= \tfrac{\pi}{12} + \tfrac{\pi}{4}$$

$$= \tfrac{\pi}{3} \quad (5pts)$$